Math 431 Math Engineers & Sci I Summer 2025

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1.1 Laplace Transforms

Let f(t) be defined for $t \geq 0$. Its Laplace transform is defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt$$

for any s that the integral converges.

Sometimes we use L(f) = F(s) = $\int_0^\infty e^{-st}f(t)dt$ and f(t) is said to be the inverse Laplace transform of F(s) or L(f). Written as $f(t) = L^{-1}(F(s))$.

Example: Find the laplace transform of $f(t) = cos(\omega t)$ where omega is a constant.

$$\mathcal{L}\{cos(\omega t)\} = \int_0^\infty e^{-st}cos(\omega t)dt = \lim_{b\to\infty}\int_0^b e^{-st}cos(\omega t)dt$$

For the rest of the integrals, I won't be 100 percent accurate. You should put limit as b goes to infinity.

Using integration by parts ($\int u\,dv = uv - \int v\,du$), we can solve this integral.
Let **u** = $cos(\omega t)$ and **dv** = $e^{-st}dt$.
Then, **du** = $-\omega sin(\omega t)dt$ and **v** = $-\frac{1}{s}e^{-st}$.

$$\int_0^b e^{-st}cos(\omega t)dt = -\frac{1}{s}e^{-st}cos(\omega t)\Big|_0^b + \frac{\omega}{s}\int_0^b e^{-st}sin(\omega t)dt \tag{1}$$

we use IBP again on the second integral.

$$\int_0^b e^{-st}sin(\omega t)dt = -\frac{1}{s}e^{-st}sin(\omega t)\Big|_0^b - \frac{\omega}{s}\int_0^b e^{-st}cos(\omega t)dt$$

substituting this back into the first integral, we have

$$\int_0^b e^{-st}cos(\omega t)dt = -\frac{1}{s}e^{-st}cos(\omega t) + \frac{\omega}{s}\left(-\frac{1}{s}e^{-st}sin(\omega t) - \frac{\omega}{s}\int_0^b e^{-st}cos(\omega t)dt\right) \tag{1}$$

$$= -\frac{1}{s}e^{-st}cos(\omega t)\Big|_0^b - \frac{\omega}{s^2}e^{-st}sin(\omega t)\Big|_0^b - \frac{\omega^2}{s^2}\int_0^b e^{-st}cos(\omega t)dt$$

Add $\frac{\omega^2}{s^2}\int_0^b e^{-st}cos(\omega t)dt$ to both sides:

$$\left(1+\frac{\omega^2}{s^2}\right)\int_0^b e^{-st}cos(\omega t)dt = -\frac{1}{s}e^{-st}cos(\omega t)\Big|_0^b - \frac{\omega}{s^2}e^{-st}sin(\omega t)\Big|_0^b = \frac{s^2+\omega^2}{s^2}\int_0^b e^{-st}cos(\omega t)dt$$

$-\frac{1}{s}e^{-st}cos(\omega t)\Big|_0^b$ as b goes to inf $= 0 - (-\frac{1}{s}) = \frac{1}{s}$
The sin function evaluates to $0 - 0 = 0$.
Now,

$$\frac{1}{s} = \frac{s^2+w^2}{s^2}\int_0^b e^{-st}cos(\omega t)$$

So,

$$\mathcal{L}(cos(\omega t)) = \frac{s}{s^2+w^2}$$

For the reader, you probably only need to know the formula. However, you should be able to manually solve for the laplace integral.

Also for the reader, s is a magical variable. It's often a placeholder.

For the Laplace integral of $sin(\omega t)$, we follow the same methodology except we have the laplace integral of cos already.
Let $u = sin(\omega t)$ and $dv = e^{-st}dt$.

Then, $du = \omega cos(\omega t)dt$ and $v = -\frac{1}{s}e^{-st}$.

Now, when you IBP, use the cos laplace integral to not perform so many operations.
Formula : $\mathcal{L}(sin(\omega t)) = \frac{\omega}{s^2+\omega^2}$

Example:
$L(cos(2t)) = \frac{s}{s^2+4}$
$L^{-1}(\frac{s}{s^2+3^2}) = cos(3t)$

$\mathcal{L}(sin(\omega t)) = \frac{w}{s^2+w^2}$ Some examples with plugging in:
Example: Find the laplace transform of $f(t) = e^{at}$ where a is a constant.

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}e^{at}dt = \int_0^\infty e^{(a-s)t}dt$$

This integral converges for $s > a$, so we assume $s > a$.

$$\int_0^\infty e^{(a-s)t}dt = \lim_{b\to\infty}\int_0^b e^{(a-s)t}dt = \lim_{b\to\infty}\left[\frac{e^{(a-s)t}}{a-s}\right]_0^b$$

$\lim_{b\to\infty} \frac{e^{(a-s)b}}{a-s} - \frac{e^{(a-s)0}}{(a-s)} = 0 - \frac{1}{(a-s)} = \frac{1}{(s-a)}$

$$= \frac{1}{s-a} = \mathcal{L}(e^{at})$$

Theorem (Linearality): For any two contacts α, β and any 2 functions $f(t), g(t)$,
$L(\{\alpha f(t) + \beta g(t)\}) = \alpha L(f(t)) + \beta L(g(t))$
Proof follows from the definition of the Laplace transform. Integral of a sum is the sum of the integrals.

$$\int_0^\infty e^{-st}(\alpha f(t)+\beta g(t))dt = \int_0^\infty (e^{-st}\alpha f(t)+e^{-st}\beta g(t))dt = \int_0^\infty (e^{-st}\alpha f(t))dt+\int_0^\infty e^{-st}\beta g(t)dt = \alpha L(f(t))+\beta L(g(t))$$

The Inverse Laplace transform is also Linear. that is,

$$L^{-1}(\alpha F(s) + \beta G(s)) = \alpha L^{-1}(F(s)) + \beta L^{-1}(G(s))$$

We use the linearity of the laplace transform and "factor out" the laplace function and use $L^{-1}L = I$ yeah.

$$L^{-1}(\alpha F(s) + \beta G(s)) = L^{-1}(L(\alpha f(s) + \beta g(s))) = \alpha f(s) + \beta g(s) = \alpha L^{-1}(F(s)) + \beta L^{-1}(G(s))$$

Example: Find the laplace transform of sinh(t) recall sinht $= \frac{e^t - e^{-t}}{2}$

$$L(sinh(t)) = L(\frac{e^t - e^{-t}}{2}) = \frac{1}{2}(L(e^t) - L(e^{-t}))$$

by $L(e^{at}) = \frac{1}{(s-a)}$,

$$L(sinh(t)) = \frac{1}{2}(\frac{1}{s-1} - \frac{1}{s+1}) = \frac{1}{2}\left(\frac{s+1-(s-1)}{(s-1)(s+1)}\right) = \frac{1}{2}\left(\frac{2}{s^2-1}\right) = \frac{1}{s^2-1} = \frac{1}{(s^2-1)}$$

The general solution is in the book,

$$L(\sinh(\alpha t)) = \frac{\alpha}{s^2 - \alpha^2}$$

Example: Find the laplace transform of $F(s) = \frac{s+1}{(s^2-s-6)}$
The partial fraction decomposition is

$$\frac{s+1}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$$

$A = 4/5, B = 1/5.$
$F(s) = \frac{4}{5}\frac{1}{(s-3)} + \frac{1}{5}\frac{1}{(s+2)}$

$$L^{-1}(F(s)) = \frac{4}{5}L^{-1}(\frac{1}{s-3}) + \frac{1}{5}L^{-1}(\frac{1}{s+2})$$

Using the formula $L^{-1}(\frac{1}{s-a}) = e^{at}$, we have

$$L^{-1}(F(s)) = \frac{4}{5}e^{3t} + \frac{1}{5}e^{-2t}$$

Theorem (s-shifting) : Suppose L(f) = F(s)

Then, for any constant a,

$$L(e^{at}f(t)) = F(s-a)$$

– shift in s There is a proof here, but he went through it quickly.

Example:

$$L(cos(2t)) = \frac{s}{s^2+4}L(e^{3t}cos(2t)) = \frac{(s-3)}{(s-3)^2+4}$$

1.2 Chapter 6 Section 2

: There is some confusion here, the professor said "suppose $f(t) \leq Me^{at}$" and that implies the conclusion in our theorem, but the book doesn't agree. Hopefully this nuance won't be tested.

Theorem:
$L(f') = sL(f) - f(0)$
$L(f'') = s^2L(f) - sf(0) - f'(0)$

Proof: $L(f') = \lim_{b\to\infty}\int_0^b f'e^{-st}dt = \lim_{b\to\infty}\int_0^b f'e^{-st}dt$
IBP $\lim_{b\to\infty}(uv\big|_0^b - \int_0^b(-s)e^{-st}f(t)dt)$
$limbtoinff(b)e^{-st} + sint0tobf(t)e^{-st}dt$ since f(b)

Theorem for nth derivative of a function using laplace transform:

$$L(f^{(n)}) = s^nL(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{n-1}(0)$$

For a constant n, $L(t^n) = \frac{n!}{s^(n+1)}$ using the formula from the derivatives.

Example: Find the Laplace tranform of $f(t) = tcos(3t)$, $f(0) = 0$. Then, calculate the derivatives. Use the laplace equations for the second and first derivative. WARNING: I can't see it, but there is some calculation mishap here

$$L(f'') = s^2L(f) - sf(0) - f'(0)$$

$$-6L(sin3t) - 9L(tcos3t) = s^2L(tcos3t) - 1 = \frac{3}{s^2+3^2} - 9L(tcos3t)$$

We come to

$$1 - \frac{18}{s^2+9} = (s^2+9)L(tcos(3t))$$

$$L(tcost3t) = \frac{s^2-9}{(s^2+9)^2}$$

Given $f(t), letg(t) = \int_0^t f(x)dx$ Then $g'(t) = f(t), g(0) = 0$. by $L(g') = sL(g) - g(0)$, we have $L(f) = sL(\int 0totf(x)dx)$

Theorem(transform of an integral) $L(\int_0^t f(x)dx) = \frac{1}{s}L(f)$ or $L^{-1}(1/sL(f)) = \int_0^t f(x)dx$

Example, find the inverse laplace transform of $F(s) = \frac{1}{s^3-s^2} = \frac{1}{s} \times \frac{1}{s} \times \frac{1}{s-1}$ $L^{-1}(1/s-1) = e^t$ so, $L^{-1}(1/s(s-1)) = \int_0^t e^x dx = e^t - 1$ so, the inverse laplace transform of $F(s) = 1/(s^3-s^2)$ is $\int_0^t e^x - 1dt = e^t - t - 1$.

Use laplace transform to solve $y'' + ay' + by = r(t)$. $y(0) = k0, y'(0) = k1$

assuming L(r) and L(y) exist,
apply transforms to both sides.

$$L(y'') + aL(y') + bL(y) = L(r(t))$$
$$s^2L(y) - sy(0) - y'(0) + a(sL(y) - y(0)) + bL(y) = L(r(t)) = R(s)$$
$$L(y)(s^2 + as + b) - k = s - k1 - ak0 = R(s)$$
$$L(y) = (k0s + k1 + ak0 + R(s))/(s^2 + as + b)$$

solution is $y = L^{-1}(L(y)) = L^{-1}((k0s + k1 + ak0 + R(s))/(s^2 + as + b))$
Example: Use Laplace transform to solve an IVP:

$$y'' - 4y' + 3y = t, \quad y(0) = 1, \quad y'(0) = 2$$

Apply the Laplace transform to the ODE:

$$\mathcal{L}(y'') - 4\mathcal{L}(y') + 3\mathcal{L}(y) = \mathcal{L}(t)$$

Using Laplace properties:

$$s^2\mathcal{L}(y) - sy(0) - y'(0) - 4\left(s\mathcal{L}(y) - y(0)\right) + 3\mathcal{L}(y) = \frac{1}{s^2}$$

Substitute initial values $y(0) = 1$, $y'(0) = 2$:

$$(s^2 - 4s + 3)\mathcal{L}(y) - s - 2 + 4 = \frac{1}{s^2}$$

$$(s-1)(s-3)\mathcal{L}(y) = s - 2 + \frac{1}{s^2}$$

$$\mathcal{L}(y) = \frac{s-2}{(s-1)(s-3)} + \frac{1}{s^2(s-1)(s-3)}$$

$$\mathcal{L}(y) = \frac{s^2(s-2)+1}{s^2(s-1)(s-3)}$$

Use partial fraction decomposition, $A/S + B/S^2 + C/s - 1 + D/s - 3$

$$A = 4/9B = 1/3, C = 0, D = 5/9$$

Hence,

$$L(y) = 4/9 * 1/s + 1/3 * 1/s^2 + 5/9 * 1/(s-3)$$

For the solution to the IVP,

$$y = L^{-1}(L(y)) = 4/9 * 1 + 1/3 * t + 5/9 * e^{3t}$$

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2.1 Chapter 6 Section 2

Use laplace transform to solve the IVP:

$$y'' + ay' + by = r(t) \tag{1}$$

such that

$$y(0) = k0, y'(0) = k1$$

Here, we consider the initial conditions at some arbitrary point t_0,

$$y(t_0) = k_0, y'(t_0) = k_1$$

. Let

$$\tilde{y}(t) = y(t + t_0)$$

, then

$$\tilde{y}(0) = y(t0) = k0\tilde{y}'(0) = y'(t0) = k1$$

.

Write (1) as

$$y''(t + t0) + ay(t + t0) + by(t + t0) = r(t + t0)$$

so

$$\tilde{y}'' + a\tilde{y}' + b\tilde{y} = r(t + t0)$$

Example: Solve the IVP:

$$y'' + y = 2t, y(\pi/4) = \pi/2, y'(\pi/4) = 2 - \sqrt{2}$$

Let

$$\tilde{y}(t) = y(t + \pi/4), \tilde{y}(0) = y(\pi/4) = \pi/2, \tilde{y}'(0) = y'(\pi/4) = 2 - \sqrt{2}.$$

And write the ODE as

$$y''(t + \pi/4) + y(t + \pi/4) = 2(t + \pi/4)$$

$$\tilde{y}''(t) + \tilde{y}(t) = 2(t + \pi/4)$$

Now apply the Laplace transform to the equation:

$$L(\tilde{y}''(t)) + L(\tilde{y}(t)) = L(2(t + \pi/4))$$

$$s^2 L(\tilde{y}(t)) - s\tilde{y}(0) - \tilde{y}'(0) + L(\tilde{y}(t)) = 2L(t + \pi/4) = 2\left(\frac{1}{s^2} + pi/2 * \frac{1}{s}\right)$$

$$L(\tilde{y}) = \frac{1}{s^2 + 1}(\frac{pi}{2}s + (2 - \sqrt{2}) + 2/s^2 + pi/2 * 1/s)$$

using partial fraction decomposition, we have

$$L(\tilde{y}) = -\frac{\sqrt{2}}{s^2 + 1} + \frac{2}{s^2} + \frac{\frac{\pi}{2}}{s}$$

$$\tilde{y} = L^-1L(\tilde{y}) = pi/2 + 2t - \sqrt{2}sin(t)$$

$$y(t) = \tilde{y}(t - \pi/4) = pi/2 + 2(t - \pi/4) - \sqrt{2}sin(t - \pi/4) = 2t - \sqrt{2}sin(t - \pi/4)$$

He tells us to memorize the Laplace transforms of the following functions: $cos, sin, e^at, t^n = n!/s^{n+1}$ Cos, sin, e^{at} are in the book, but t^n is not so I put it here.

2.2 Chapter 6 Section 3

6.3 Unit step function We graphed two functions, one with a shift in x. This is from a high school algebra course. Look up something like "shift in x".

Let $\tilde{f}(t - a) = f(t - a)$ for $t >= a$, 0 otherwise.

$$\begin{cases} f(t-a) & t \geq a \\ 0 & t < a \end{cases}$$

Introduce the unit step function:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$\tilde{f}(t - a) = f(t - a)u(t - a)$

Theorem: If $L(f(t)) = F(s)$, then $L(t-a)u(t-a) = e^{-as}F(s)$ $L^{-1}(e^{-as}F(s)) = f(t-a)u(t-a)$
Proof: Given

$$F(s) = \int_0^\infty e^{-st} f(t)dt$$

, then

$$L(f(t-a)u(t-a)) = \int_0^\infty e^{-st} f(t-a)u(t-a)dt$$

Since

$$u(t-a) = 1 \text{for} t \geq a$$

, and

$$0 \text{if} t < a$$

$F(s) = \int_a^\infty e^{-st} f(t-a)dt$
Let $\tau = t - a$, then $dt = d\tau, and t = \tau + a$
$= \int_0^\infty e^{-s(\tau+a)} f(\tau)d\tau = e^{-as} \int_0^\infty e^{-s\tau} f(\tau)d\tau = e^{-as}F(s)$
Example: Find the Laplace transform of

$$f(t) = \begin{cases} \cos(2t), & 0 < t < \pi \\ 0, & t \geq \pi \end{cases}$$

He writes it in piecewise form.
write $f(t) = cos(2t)(1 - u(t-\pi)) = cos(2t)u(t) - cos(2t)u(t-\pi) = cos2t - cos(2t - 2pi)u(t-\pi) = cos(2t) - cos(2(t-\pi))u(t-\pi)$
This is kinda unintuitive. To the reader, you use the unit step function to "turn off" the function after a certain point.
Since $L(cos(2t)) = s/(s^2+4)$, $L(f(t)) = L(cos(2t)) - L(cos(2(t-\pi))u(t-\pi)) = s/(s^2+4) - e^{-\pi s} * s/(s^2+4)$ done.
Example: Let

$$F(s) = \frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2+1}$$

To find the inverse Laplace,

$$L^{-1}(1/s^2) = t, L^{-1}(1/(s^2+1)) = sin(t)$$

By the result in the theorem,

$$L^{-1}(F(s)) = (t-1)u(t-1) + sin(t-2)u(t-2) = 0 \text{when} 0 < t < 1, t-1 when 1 < t < 2, and sin(t-2) + (t-1) when t > 2.$$

$$L^{-1}(F(s)) = \begin{cases} 0, & 0 < t < 1 \\ t-1, & 1 \leq t < 2 \\ (t-1) + \sin(t-2), & t \geq 2 \end{cases}$$

2.3 Chapter 7 Section 3

Consider a system of m equations with n unknowns. see the book.
Where $x_1, x_2, \ldots, x_n$ are the unknowns and aij are the coefficients and given constants.
Let A be the matrix of coefficients, x the vector of unknowns, and b the vector of constants.
Then the system is expressed as AX = b.
The augmented matrix is the matrix formed by appending the vector b to the matrix A.
Example: write the augmented matrix of the system:

$$\begin{cases} 2x_1 - 4x_2 - 4x_3 + x_4 = -5 \\ -4x_1 + 6x_3 - 3x_4 = 7 \\ 2x_1 + 5x_2 - 7x_3 = 9 \end{cases}$$

$$\begin{bmatrix} 3 & -4 & -4 & 1 & -5 \\ -4 & 0 & 6 & -3 & 7 \\ 2 & 5 & -7 & 0 & 9 \end{bmatrix}$$

Notice the 0 when a value for x_k is not present.

Row operations: 1) Interchange two rows.
2) Multiply row by a nonzero constant
3) Add a multiple of one row to another row.

Example: Use row operations to solve the system:

$$\begin{cases} x + y - z = 1 \\ x - y + 2z = 2 \\ -x + 3y + z = 3 \end{cases}$$

We write the augmented matrix:

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & 2 \\ -1 & 3 & 1 & 3 \end{bmatrix}$$

Step 1:

$$\begin{matrix} 1 & 1 & -1 & 1 & - \\ 0 & -2 & 3 & 1 & R2 - R1 \\ 0 & 4 & 0 & 4 & R3 + R1 \end{matrix}$$

Step 2: If possible, change column 2 into 0 1 0 without alterning column 1 0 0.

$$\begin{matrix} 1 & 1 & -1 & 1 & - \\ 0 & -2 & 3 & 1 & - \\ 0 & 1 & 0 & 1 & (\frac{1}{4}R3) \end{matrix}$$

.

$$\begin{matrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 3 & 1 \end{matrix}$$

.

.

$$\begin{matrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 3 \end{matrix}$$

Step 3: Change column 3 in to 0 0 1 without altering (1 0 0) , (0 1 0) if possible.

$$\begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{matrix}$$

The solution is x = 1, y = 1, z = 1.

Any linear equation in x,y and t.

$$ax + xy + ct = d$$

represents a plane in 3D space.

Consider a linear system of 3 equations with 3 unknowns

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{cases}$$

If 3 planes intersect at a point, then we have a solution that is **unique**. That is, x , y , and z don't depend on each other in the solution. If the planes are parallel, then there is no solution. If the planes intersect in a line, then there are infinitely many solutions and those solutions are a line in 3d space

$$\begin{cases} x + y + z = 1 \\ 2x + 4y + z = 1 \\ -x + 2y + 2z = 2 \end{cases}$$

the augmented matrix is:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 1 & 1 \\ -1 & 2 & 2 & -2 \end{bmatrix}$$

Step 1:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 3 & 3 & -1 \end{bmatrix}$$

step 2:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & -1 \end{bmatrix}$$

step 3:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 4/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/3 \end{bmatrix}$$

because x,y,z don't rely on each other (you can think of it as a point), we have a unique solution.

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A little help : I don't think I was clear enough about Laplace Transforms. It's simply used to solve differential equations. s is an intermediary number we use to perform the transform. Exercises are your best friend for how to use it. These notes will only get you so far.

Review Matrix Arithmetic :
Use Gauss Elimination to solve

$$\begin{cases} x + y - z = 3 \\ 2x - y + 2z = 5 \\ -x + y + 2z = 1 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & -1 & 2 & 5 \\ -1 & 1 & 2 & 1 \end{bmatrix}$$

Okay, follow the steps from last lecture, or go **HERE** for the calculation (click the details dropdown). x = 2, y = 1, z = 1. To verify, plug in the values into the original equations.

Example:

$$\begin{cases} x - 2y + z = 1 \\ x + y + z = 6 \\ 3x - 4y + 3z = 0 \end{cases}$$

Augmented Matrix:

$$\begin{bmatrix} 1 & -2 & 1 & 1 \\ 1 & 1 & 1 & 6 \\ 3 & -4 & 3 & 0 \end{bmatrix}$$

Operations:

$$\begin{bmatrix} 1 & -2 & 1 & 1 & - \\ 0 & 3 & 0 & 5 & R2 - R1 \\ 0 & 2 & 0 & -3 & R3 - 3R1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & 1 & - \\ 0 & 1 & 0 & 5/3 & \text{(R2/3)} \\ 0 & 1 & 0 & -3/2 & \text{(R3/2)} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 13/3 & R1+2R2 \\ 0 & 1 & 0 & 5/3 & - \\ 0 & 0 & 0 & -19/6 & R3-R2 \end{bmatrix}$$

Is it possible to change column 3 into $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ without altering $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$? No.

$$\begin{cases} x+z=\frac{25}{3} \\ y=5/3 \\ 0=-19/6 \text{which is not true.} \end{cases}$$

So, the system (is inconsistant and) has no solution. To the reader, you could have have also seen at the second to last matrix that y = 5/3 = -3/2 which isn't true.

Example: Solve

$$\begin{cases} x+y-z=1 \\ x+2y-z=2 \\ -x+z=0 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 2 & -1 & 2 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & 1 & - \\ 0 & 1 & 0 & 1 & R2-R1 \\ 0 & 1 & 0 & 1 & R3+R1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & R1-R2 \\ 0 & 1 & 0 & 1 & - \\ 0 & 0 & 0 & 0 & - \end{bmatrix}$$

Now,

$$\begin{cases} x-z=0 \implies x=z \\ y=1 \\ 0=0 \end{cases}$$

$$x=t$$
$$y=1$$
$$z=t$$

where t is any number. Or, you could say y = 1, x=z for all values of x or z. This defines a line in 3d space.

3.1 Determinants

For a 2 by 2 matrix, the determinant of A or det(A) or

$$det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad-bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

The vertical lines is notation for determinant.

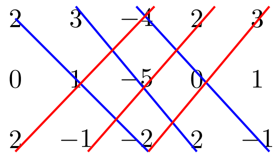
3 by 3 We append column 1 to the end of the matrix. Then append column 2 to the end of the matrix with the first appended.
Swipe down and right for the first 3 columns. Add those products.
Then, up and right for the first 3 columns again. Subtract those products.

Calculate

$$\begin{vmatrix} 2 & 3 & -4 \\ 0 & 1 & -5 \\ 2 & -1 & -2 \end{vmatrix}$$

Appending:

$$\begin{bmatrix} 2 & 3 & -4 & 2 & 3 \\ 0 & 1 & -5 & 0 & 1 \\ 2 & -1 & -2 & 2 & -1 \end{bmatrix}$$



$$2 \times 1 \times (-2) + 3 \times (-5) \times 2 + (-4) \times 0 \times (-1)$$
$$-(2 \times 1 \times (-4) + (-1) \times (-5) \times 2 + (-2) \times 0 \times 3)$$
$$= -4 + 15 + 8 - (-8 + 15 + 8) = 4$$

3.2 Determinants of higher dimensionality (and in general)

:
The **minor** of an entry a_{ij} is the determinant obtained by the matrix after deleting the i and j column and row respectively.

$M_{ij} =$

$$\begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

Notice that there are no entries for $a_{i,-}$ or $a_{-,j}$.
We define the **cofactor** to be $= A_{ij} = (-1)^{i+j} M_{ij}$ where M is the determinant of the minor.
Example: Let det(A) =

$$\begin{vmatrix} -3 & -4 & 5 \\ 6 & 7 & 0 \\ 2 & -3 & 1 \end{vmatrix}$$

$a_{1,3} = 5$ $a_{2,1} = 6$ $a_{3,2} = -3$

For $a_{1,3} = 5, M_{1,3} = \begin{vmatrix} 6 & 7 \\ 2 & -3 \end{vmatrix} = 6 * (-3) - 2 * 7 = -32$ and $A_{1,3} = (-1)^{1+3} * -32 = 1 * (-32) = -32$

The professor does the same for $a_{2,1}$ and $a_{3,2}$.

Along the ith row, the detminant is

$$a_{i,1}A_{i,1} + a_{i,2}Ai2 + \cdots + a_{i,n}A_{i,n}$$

and along the jth column,

$$a_{1,j}A_{1,j} + a_{2,j}A2j + \cdots + a_{n,j}A_{n,j}$$

.

Note: **This is a new way to calculate the determinant, just go down either a column or a row and calculate cofactors. YOU CAN GO DOWN ANY ROW/COLUMN.**

Example: Find det(A) =

$$\begin{vmatrix} 2 & 3 & -1 \\ 4 & 0 & 2 \\ -1 & 2 & 3 \end{vmatrix}$$

Append the first columns as the first methodology specifies

$$\begin{bmatrix} 2 & 3 & -1 & 2 & 3 \\ 4 & 0 & 2 & 4 & 0 \\ -1 & 2 & 3 & -1 & 2 \end{bmatrix}$$

$$0 - 6 - 8 - 0 - 8 - 36 = \boxed{-58}$$

Along the second row cofactor expansion.

$$\begin{vmatrix} 2 & 3 & -1 \\ 4 & 0 & 2 \\ -1 & 2 & 3 \end{vmatrix} = 4 \cdot A_{21} + 0 \cdot A_{22} + 2 \cdot A_{23}$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix} = -1 \cdot (3 \cdot 3 - (-1) \cdot 2) = -1 \cdot (9 + 2) = -11$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} = -1 \cdot (2 \cdot 2 - 3 \cdot (-1)) = -1 \cdot (4 + 3) = -7$$

$$\det = 4 \cdot (-11) + 0 \cdot A_{22} + 2 \cdot (-7) = -44 + 0 - 14 = \boxed{-58}$$

Along the third column cofactor expansion

$$\begin{vmatrix} 2 & 3 & -1 \\ 4 & 0 & 2 \\ -1 & 2 & 3 \end{vmatrix} = (-1) \cdot A_{13} + 2 \cdot A_{23} + 3 \cdot A_{33}$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 4 & 0 \\ -1 & 2 \end{vmatrix} = 1 \cdot (4 \cdot 2 - 0 \cdot (-1)) = 8$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} = -1 \cdot (2 \cdot 2 - 3 \cdot (-1)) = -1 \cdot (4 + 3) = -7$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix} = 1 \cdot (2 \cdot 0 - 3 \cdot 4) = -12$$

$$\det = (-1) \cdot 8 + 2 \cdot (-7) + 3 \cdot (-12) = -8 - 14 - 36 = \boxed{-58}$$

I think it's a good **exercise** write out the operations using arbitrary matrices to understand why they are the same operation.

Example 4x4 determinant: det(A) = (going down the first column to calculate the determinant)

$$\begin{vmatrix} 1 & -1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 3 & 0 & 1 & -1 \\ 0 & 2 & 0 & 3 \end{vmatrix}$$

det(A) =

$$1 \cdot (-1)^{1+1=2} \cdot \begin{vmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 3 \end{vmatrix} + 0 + 3 \cdot (-1)^{3+1=4} \begin{vmatrix} -1 & 0 & -1 \\ 2 & 1 & 0 \\ 2 & 0 & 3 \end{vmatrix} + 0$$

$$= (2 \cdot (-1)^{1+1=2} \cdot \begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} + 0 + 2 \cdot (-1)^{3+1=4} \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix}) + 3(1 \cdot (-1)^{2+2=4} \begin{vmatrix} -1 & 1 \\ 2 & 3 \end{vmatrix})$$

Notice our choice of which column to use. In particular, for the second 3x3 matrix, we used the middle column. Because there are 2 zeros, we only need to perform one determinant/cofactor calculation. (see 3.2)

$$= 2 \cdot 3 + 2 \cdot (-1) + 3 \cdot (-3 - 2) = 6 - 2 - 15 = -11$$

(professor got wrong)

Next lesson: Row and Column operations can aid in simplifying the process of calculating determinants. Recall: What have we done to matrices to make then "simpler"?

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4.1 Review Determinants

For a 3x3 matrix A:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

$$Det(A) = a_{1,1} \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2} \begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3} \begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}$$

which is cofactor expansion of row 1

$$\begin{vmatrix} 3 & 1 & 2 \\ -1 & 0 & 4 \\ 5 & 1 & 6 \end{vmatrix}$$

(a) using above calculation
(b) along column 2
(c) along row 3

(a) Cofactor expansion along row 1

$$\begin{aligned} \det(A) &= 3 \begin{vmatrix} 0 & 4 \\ 1 & 6 \end{vmatrix} - 1 \begin{vmatrix} -1 & 4 \\ 5 & 6 \end{vmatrix} + 2 \begin{vmatrix} -1 & 0 \\ 5 & 1 \end{vmatrix} \\ &= 3(0 \cdot 6 - 4 \cdot 1) - 1((-1)(6) - (4)(5)) + 2((-1)(1) - (0)(5)) \\ &= 3(-4) - 1(-6 - 20) + 2(-1) \\ &= -12 + 26 - 2 = \boxed{12} \end{aligned}$$

(b) Cofactor expansion along column 2

$$\begin{aligned} \det(A) &= -1^{1+2} \cdot 1 \cdot \begin{vmatrix} -1 & 4 \\ 5 & 6 \end{vmatrix} + (-1)^{2+2} \cdot 0 \cdot \begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} + (-1)^{3+2} \cdot 1 \cdot \begin{vmatrix} 3 & 2 \\ -1 & 4 \end{vmatrix} \\ &= -1 \cdot (-26) + 0 - 1 \cdot 14 \\ &= 26 - 14 = \boxed{12} \end{aligned}$$

(c) Cofactor expansion along row 3

$$\begin{aligned}
\det(A) &= (-1)^{3+1} \cdot 5 \cdot \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} + (-1)^{3+2} \cdot 1 \cdot \begin{vmatrix} 3 & 2 \\ -1 & 4 \end{vmatrix} + (-1)^{3+3} \cdot 6 \cdot \begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix} \\
&= (+1) \cdot 5 \cdot (1 \cdot 4 - 2 \cdot 0) - 1 \cdot (3 \cdot 4 - 2 \cdot (-1)) + 1 \cdot 6 \cdot (3 \cdot 0 - 1 \cdot (-1)) \\
&= 5(4) - 14 + 6(1) = 20 - 14 + 6 = \boxed{12}
\end{aligned}$$

Example find det:

$$\begin{vmatrix} 2 & 1 & 0 & 1 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & -1 & 2 \\ 3 & 0 & 2 & 0 \end{vmatrix}$$

Choose the row or column with the most zeros. **Column 2 is ideal**

$$\begin{aligned}
Det = (-1)^{1+2} \cdot 1 \cdot \begin{vmatrix} -1 & 3 & 1 \\ 0 & -1 & 2 \\ 3 & 2 & 0 \end{vmatrix} + (-1)^{2+2} \cdot 0 \cdot \begin{vmatrix} 2 & 0 & 1 \\ 0 & -1 & 2 \\ 3 & 2 & 0 \end{vmatrix} \\
+ (-1)^{3+2} \cdot 1 \cdot \begin{vmatrix} 2 & 0 & 1 \\ -1 & 3 & 1 \\ 3 & 2 & 0 \end{vmatrix} + (-1)^{4+2} \cdot 0 \cdot \begin{vmatrix} 2 & 0 & 1 \\ -1 & 3 & 1 \\ 0 & -1 & 2 \end{vmatrix}
\end{aligned}$$

$$Det = -1 \cdot \begin{vmatrix} -1 & 3 & 1 \\ 0 & -1 & 2 \\ 3 & 2 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & 0 & 1 \\ -1 & 3 & 1 \\ 3 & 2 & 0 \end{vmatrix} = D1 + D2$$

$$D_1 = -1 \begin{vmatrix} -1 & 3 & 1 \\ 0 & -1 & 2 \\ 3 & 2 & 0 \end{vmatrix} = -25 \text{by whatever method you want}$$

$$\begin{aligned}
\begin{vmatrix} 2 & 0 & 1 \\ -1 & 3 & 1 \\ 3 & 2 & 0 \end{vmatrix} &= 1 \cdot (-1)^{1+3} \begin{vmatrix} -1 & 3 \\ 3 & 2 \end{vmatrix} + 1 \cdot (-1)^{2+3} \begin{vmatrix} 2 & 0 \\ 3 & 2 \end{vmatrix} + 0 \cdot (-1)^{3+3} \begin{vmatrix} 2 & 0 \\ -1 & 3 \end{vmatrix} \\
&= 1 \cdot (1) \left[(-1)(2) - (3)(3)\right] + 1 \cdot (-1) \left[(2)(2) - (0)(3)\right] + 0 \cdot \left[(2)(3) - (0)(-1)\right] \\
&= 1 \cdot (-2 - 9) - (4) + 0 \\
&= (-11) - 4 \\
&= -15
\end{aligned}$$

Notice the choice for columns in the 3 by 3 case, you should choose the columns/rows with the most zeros. For the second determinant, we chose the thrid column because of its zero.

$D_2 = (-1) * -15 = 15$

$D_1 + D_2 = -10$ is the det.

Theorem (Properties of Determinants): Let A be an $n \times n$ matrix, and let B be obtained from A by a single elementary row or column operation.

(a) **Row/Column Interchange:** If B is obtained by interchanging two rows (or columns) of A, then $\det(B) = -\det(A)$.

(b) **Row/Column Scaling:** If B is obtained by multiplying a row (or column) of A by a nonzero constant c, then $\det(B) = c \cdot \det(A)$.

(c) **Row/Column Replacement:** If B is obtained by adding a multiple of one row (or column) to another row (or column), then $\det(B) = \det(A)$.

Example: Let

$$A = \begin{bmatrix} 3 & -4 \\ 6 & 5 \end{bmatrix}$$

Doing operations on A to get B, we get: Interchange/swap

$$B = \begin{bmatrix} 6 & 5 \\ 3 & -4 \end{bmatrix}$$

$$\det(A) = 3 \cdot 5 - 6 \cdot (-4) = 15 + 24 = 39$$
$$\det(B) = 6 \cdot (-4) - 5 \cdot 3 = -24 - 15 = -39$$

Scaling first column

$$B = \begin{bmatrix} -9 & -4 \\ -18 & 5 \end{bmatrix}$$
$$\det(B) = (-9) \cdot 5 - (-18) \cdot (-4) = -45 - 72 = -117 = -3 \cdot 39$$

Adding 2 times the first row to the second.

$$B = \begin{bmatrix} 3 & -4 \\ 12 & -3 \end{bmatrix}$$

$$\det(B) = 3 \cdot (-3) - 12 \cdot (-4) = -9 + 48 = 39$$

Example :

$$\begin{vmatrix} 2 & -1 & 1 & 2 \\ 3 & 2 & -1 & 0 \\ 0 & -3 & 1 & 2 \\ -1 & 1 & 0 & 1 \end{vmatrix}$$

Use row or column operations to make a row or column with only one nonzero entry and use the cofactor expansion.

$$\begin{vmatrix} 2 & -1 & 1 & 2 & - \\ 5 & 1 & 0 & 2 & R2 + R1 \\ -2 & -2 & 0 & 0 & R3 - R1 \\ -1 & 1 & 0 & 1 & - \end{vmatrix}$$

Expand along the third column (which has only one nonzero entry):

$$1 \cdot (-1)^{1+3} \begin{vmatrix} 5 & 1 & 2 \\ -2 & -2 & 0 \\ -1 & 1 & 1 \end{vmatrix}$$

Now, subtract column 1 from column 2 in the 3×3 matrix:

$$\begin{bmatrix} 5 & -4 & 2 \\ -2 & 0 & 0 \\ -1 & 2 & 1 \\ - & C2 - C1 & - \end{bmatrix}$$

Now, use row 2 for cofactor expansion (since it has two zeros).
$determinant = -16$

Example: Find the determinant:

$$\begin{vmatrix} 2 & -1 & 1 & 0 & 3 \\ -1 & 2 & 3 & 1 & -1 \\ 0 & 2 & -1 & 0 & 1 \\ 4 & -1 & 2 & 1 & 0 \\ -1 & 2 & -1 & -1 & 0 \end{vmatrix}$$

Column 4 is crucial because it can be turned into a column of zeros (except for one entry) using row operations.

Apply row operations:

$$\begin{bmatrix} 2 & -1 & 1 & 0 & 3 & - \\ -1 & 2 & 3 & 1 & -1 & - \\ 0 & 2 & -1 & 0 & 1 & - \\ 5 & -3 & -1 & 0 & 1 & R4-R2 \\ -2 & 4 & 2 & 0 & -1 & R5+R2 \end{bmatrix}$$

Now expand along column 4 (which has only one nonzero entry in row 2):

$$1 \cdot (-1)^{2+4} \begin{vmatrix} 2 & -7 & 4 & 3 \\ 0 & 0 & 0 & 1 \\ 5 & -5 & 0 & 1 \\ -2 & 5 & 1 & 7 \end{vmatrix}$$

(Here, columns 2 and 3 have been replaced by $C_2 - C_4$ and $C_3 - C_4$ for simplification.)

Now, expand along the second row (which has three zeros):

$$0 + 0 + 0 + 1 \cdot (-1)^{2+4} \begin{vmatrix} 2 & -5 & 4 \\ 5 & 0 & 0 \\ -2 & 3 & 1 \end{vmatrix}$$

(Here, $C_2 \leftarrow C_2 + C_1$ in the 3 by 3)

Now, expand along the second row (which has two zeros):

$$= 5 \cdot \begin{vmatrix} -5 & 4 \\ 4 & 1 \end{vmatrix} = 5\left((-5)(1) - (4)(4)\right) = 5(-5-16) = 5(-21) = -105$$

So, the determinant is $\boxed{-105}$ **I think the teacher flipped a sign. The answer is 105, not -105.**. **YES! 5 is supposed to be -5 in the above calculation from** $5 * (-1)^{1+2=3} = -5$ Brilliant.

4.2 Cramer's Rule

Given a linear system of n equations in n unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

Let A be the coefficient matrix, and let $D = \det(A)$. If $D \neq 0$, the system has a unique solution given by

$$x_k = \frac{D_k}{D}, \quad \text{for } k = 1, 2, \ldots, n$$

where D_k is the determinant of the matrix obtained from A by replacing its k-th column with the column vector $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$.

Example: Use cramer's rule to solve

$$\begin{cases} 2x - 3y = -9 \\ 3x + 4y = 5 \end{cases}$$

$$\begin{vmatrix} 2 & -3 \\ 3 & 4 \end{vmatrix} = 17 \neq 0.$$

The system has a unique solution.

$$x = \frac{D_1}{D} = \frac{\begin{vmatrix} -9 & -3 \\ 5 & 4 \end{vmatrix}}{17} = \frac{(-9)(4) - (5)(-3)}{17} = \frac{-36 + 15}{17} = \frac{-21}{17}$$

$$y = \frac{D_2}{D} = \frac{\begin{vmatrix} 2 & -9 \\ 3 & 5 \end{vmatrix}}{17} = \frac{2 \cdot 5 - 3 \cdot (-9)}{17} = \frac{10 + 27}{17} = \frac{37}{17}$$

Example : Use Cramer's rule to solve

$$\begin{cases} 2x - y + 2z = 1 \\ -x + 2y - z = 2 \\ x + 0y + 2z = -1 \end{cases}$$

The coefficient matrix is

$$A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

and the constant vector is

$$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Compute the determinant:

$$D = \begin{vmatrix} 2 & -1 & -2 \\ -1 & 2 & 1 \\ 1 & 0 & 0 \\ - & - & C_3 - 2C_1 \end{vmatrix}$$

Expand along the third row:

$$D = 1 \cdot \begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} - 0 + 0$$
$$= 3 \neq 0$$

Therefore, we have a unique solution.
Now, compute D_1, D_2, D_3 by replacing the respective columns with $\mathbf{b}$:

$$D_1 = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 0 & 2 \end{vmatrix}$$

Expand along the third row:

$$D_1 = -1 \cdot \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} + 0 \cdot \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix}$$

$$= -1 \cdot ((-1)(-1) - 2 \cdot 2) + 2 \cdot (1 \cdot 2 - (-1) \cdot 2) = -1 \cdot (1 - 4) + 2 \cdot (2 + 2) = -1 \cdot (-3) + 2 \cdot 4 = 3 + 8 = 11$$

$$D_2 = \begin{vmatrix} 2 & 1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

Expand along the third row:

$$D_2 = 1 \cdot \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 2 & 2 \\ -1 & -1 \end{vmatrix} + 2 \cdot \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}$$

$$= 1 \cdot (1 \cdot -1 - 2 \cdot 2) + 1 \cdot (2 \cdot -1 - (-1) \cdot 2) + 2 \cdot (2 \cdot 2 - (-1) \cdot 1)$$

$$= 1 \cdot (-1 - 4) + 1 \cdot (-2 + 2) + 2 \cdot (4 + 1) = 1 \cdot (-5) + 1 \cdot 0 + 2 \cdot 5 = -5 + 0 + 10 = 5$$

$$D_3 = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & 2 \\ 1 & 0 & -1 \end{vmatrix}$$

Expand along the third row:

$$D_3 = 1 \cdot \begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix} - 0 \cdot \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= 1 \cdot ((-1)(2) - 1 \cdot 2) + (-1) \cdot (2 \cdot 2 - (-1) \cdot (-1)) = 1 \cdot (-2 - 2) + (-1) \cdot (4 - 1) = 1 \cdot (-4) + (-1) \cdot 3 = -4 - 3 = -7$$

Therefore, the solution is:

$$x = \frac{D_1}{D} = \frac{11}{3}, \quad y = \frac{D_2}{D} = \frac{5}{3}, \quad z = \frac{D_3}{D} = \frac{-7}{3}$$

Honestly, you don't need to do the z calculation. You should, but it's also sufficient to substitute into the linear system.

4.3 7.4 Independence

Given n vectors $v_1, v_2, \ldots, v_n$ in the same dimension, they are said to be **linearly independent** if the equation $c_1 v_2 + c_2 v_2 + \cdots + c_n v_n = 0$ (zero vector) only holds when $c_1 = c_2 = \cdots = c_n = 0$ (0 number). I should note the 0 vector is a column of 0's unless specified otherwise.

Example : v1 = $\begin{pmatrix} 1 \\ 2 \\ 3 \\ -1 \end{pmatrix}$, v2 = $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, v3= $\begin{pmatrix} 1 \\ -2 \\ 0 \\ 3 \end{pmatrix}$. v4 = $\begin{pmatrix} 3 \\ 0 \\ 4 \\ 2 \end{pmatrix}$

Test independence.

$c_1 v_1 + c_2 v_2$ and so on $= 0$ (This is our assumption). We have to show that c_k either has to be 0 for all values of k or that it can be something else.

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 0 & -2 & 0 \\ 3 & 1 & 0 & 4 \\ -1 & 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented Matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 3 & 0 \\ 2 & 0 & -2 & 0 & 0 \\ 3 & 1 & 0 & 4 & 0 \\ -1 & 0 & 3 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 3 & 0 \\ 0 & -2 & -4 & -6 & 0 \\ 0 & -2 & -3 & -5 & 0 \\ 0 & 1 & 4 & 5 & 0 \end{bmatrix}$$

(Row operations: $R_2 - 2R_1$, $R_3 - 3R_1$, $R_4 + R_1$)

$$\begin{bmatrix} 1 & 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 2 & 3 & 5 & 0 \\ 0 & 1 & 4 & 5 & 0 \end{bmatrix}$$

(Row operations: $-\frac{1}{2}R_2$, $-R_3$)

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & 2 & 0 \end{bmatrix}$$

(Row operations: $R_1 - R_2$, $R_3 - 2R_2$, $R_4 - R_2$)

Now, R3 = -R3
R4 = 1/2 R4
Then,

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & R1 + R3 \\ 0 & 1 & 0 & 1 & 0 & R2 - 2R3 \\ 0 & 0 & 1 & 1 & 0 & - \\ 0 & 0 & 0 & 0 & 0 & R4 - R3 \end{bmatrix}$$

It's impossible to change column 4 into $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ without altering the other 3 previous columns. Now, undoing the matrix representation,

$$\begin{cases} c_1 + c_4 = 0 \\ c_2 + c_4 = 0 \\ c_3 + c_4 = 0 \\ 0 = 0 \end{cases}$$

or

$$\begin{cases} c_4 = -c_1 \\ c_4 = -c_2 \\ c_4 = -c_3 \\ 0 = 0 \end{cases}$$

$c_4 = -c_1 = -c_2 = -c_3$ for any value.

Hence, you can choose any value for the varibles. Any in this case means you have unlimited choices. Because one of the any (say 1,2, π) are not 0, we have linear **dependence**. That is, every c does not have to be 0.

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7.8

5.1 Inverse and Algorithm

Given an $n \times n$ matrix A, another $n \times n$ matrix B is said to be the inverse of A if $AB = I_n$ and $BA = I_n$ Written as $B = A^{-1}$ Example: Let A =

$$\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

Find A^{-1} if possible. Assume $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ such that $AB = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

AB = $\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$= \begin{bmatrix} 2x + z & 2y + w \\ -x + 3z & -y + 3w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equality comes from individual positions.

$$\begin{cases} 2x + z = 1 \\ -x + 3z = 0 \end{cases}$$

$$\begin{cases} 2y + w = 0 \\ -y + 3w = 1 \end{cases}$$

$$\text{(for x,z)} \begin{bmatrix} 2 & 1 & 1 \\ -1 & 3 & 0 \end{bmatrix} \quad \text{(for y,w)} \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}$$

Use the row operations we've been using.

$$\text{(for x,z)} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 7 & 1 \end{bmatrix} \quad \text{(for y,w)} \begin{bmatrix} 1 & -3 & -1 \\ 0 & 7 & 2 \end{bmatrix}$$

$$\text{(for x,z)} \begin{bmatrix} 1 & 0 & 3/7 \\ 0 & 1 & 1/7 \end{bmatrix} \quad \text{(for y,w)} \begin{bmatrix} 1 & 0 & -1/7 \\ 0 & 1 & 2/7 \end{bmatrix}$$

$$\begin{cases} x = 3/7 \\ y = -1/7 \\ z = 1/7 \\ w = 2/7 \end{cases}$$

Therefore, $\mathbf{A}^{-1} = \frac{1}{7}\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$

5.2 Inverse Algorithm 2

: Append the Identity matrix to the side of the matrix.

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ -1 & 3 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 0 & 7/2 & 1/2 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/14 & 1/7 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & 3/7 & -1/7 \\ 0 & 1 & 1/7 & 2/7 \end{array}\right)$$

You end up with $(I_2|A^{-1})$.

Example : Find the inverse of: A =

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

if possible.

$(AI_3) =$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 1 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -2 & 0 & 0 & -1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{7}{3} & \frac{1}{3} & \frac{1}{3} & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{7} & -\frac{4}{7} & \frac{2}{7} \\ 0 & 1 & 0 & \frac{2}{7} & \frac{2}{7} & -\frac{1}{7} \\ 0 & 0 & 1 & \frac{1}{7} & \frac{1}{7} & \frac{3}{7} \end{array}\right)$$

Sorry if the last calculation was fast. You multiply the bottom row by $\frac{3}{7}$ then subtract upwards using the ones in the left square matrix's diagonal.

$= (I_3|A^{-1})$

Cofactor Matrix The cofactor matrix of an $n \times n$ matrix is a $n \times n$ where each entry is instead the cofactor of the entry from the original matrix.

Adjoint of A (adj(A)) The transpose of the cofactor matrix.

Theorem Existance and Formula for inverse : An $n \times n$ matrix A has an inverse iff $det(A) \neq 0$. And in this case, $A^{-1} = \frac{1}{det(A)} *$ (Cofactor matrix)$^T = \frac{1}{det(A)} *$ (Adj(A))

$$\text{For A} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$$

If $det(A) = a_{1,1} \times a_{2,2} - a_{1,2} \times a_{2,2} \neq 0$,

$$\textbf{Cofactors} = \begin{cases} C_{1,1} = a_{2,2} \\ C_{1,2} = -a_{2,1} \\ C_{2,1} = -a_{1,2} \\ C_{2,2} = a_{1,1} \end{cases}$$

$A^{-1} = \frac{1}{det(A)} * \begin{bmatrix} a_{2,2} & -a_{2,1} \\ -a_{1,2} & a_{1,1} \end{bmatrix}^T = \frac{1}{det(A)} * \begin{bmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{bmatrix}$

Example:

$$\text{Find the inverse of A} = \begin{bmatrix} 6 & 5 \\ -5 & 7 \end{bmatrix}$$

if possible. $Det(A) = 67 \neq 0$. Thus, A^{-1} exists. By the formula, $A^{-1} = \frac{1}{67} \cdot \begin{bmatrix} 7 & -5 \\ 5 & 6 \end{bmatrix}$

Consider a linear system of n equations with n unknowns.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

Let A = the coefficient matrix of a's as we've seen before in **4.2**. $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$.

Then the system can be written as $AX = b$. If $det(A) \neq 0$, from cramer's rule, we have the unique solution

$$x_k = \frac{D_k}{D}, \quad \text{for } k = 1, 2, \ldots, n$$

where n+1 $n \times n$ determinants need to be calculated (n for every k and +1 for the original matrix).

Since $det(A) \neq 0, A^{-1}$ exists. $A^{-1}AX = A^{-1}b$

$IX = X = A^{-1}b$

$X = A^{-1}b$ ★

Example : Solve (a)

$$\begin{cases} x + 2y = 5 \\ -x + y + z = -1 \\ -y + 2z = 3 \end{cases}$$

(b)

$$\begin{cases} x + 2y = -6 \\ -x + y + z = 0 \\ -y + 2z = -2 \end{cases}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

Its inverse was computed before. 3 4 2 matrix times 5 -1 3

3 4 2 matrix times -6 0 -2

5.3 8.1

Let A be an $n \times n$ matrix. A scalar λ is said to be an eigenvalue of A if $det(A - \lambda I_n) = 0$. And correspondingly, if a vector X satisfies $(A - \lambda I_n)X = 0$ (weird notations), $AX = \lambda X$ X is called an eigenvector of A relative to the eigenvalue λ.

Example: Find eigenvalues and eigenvectors of A =

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Eigenvalues: Solve $det(A - \lambda I_3) = 0$

$0 =$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}$$

We use row/column operations to help.

See picture at 1039

$=$

$$2 * (-1)^{1+2=3} \begin{bmatrix} -(\lambda+3)(\lambda-2)/2 & -3(\lambda+3)/2 - (\lambda+3) & -(\lambda+3) \end{bmatrix}$$

Use multiply by $-1/(\lambda + 3)$ twice to get.

$$= -2 * (\lambda+3) * \begin{matrix} ((\lambda-2)/2 & 3/2 \\ 1 & 1) \end{matrix} = -2 * (\lambda+3)^2 * ((\lambda-2)/2 - 3/2) = (\lambda+3)^2(\lambda-5)$$

$$\begin{cases} \lambda_1 = 5 \\ \lambda_2 = \lambda_3 = -3. \end{cases}$$

$(A - 5I_3)X = 0$

$$\begin{bmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & 0-5 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}$$

Make the augmented matrix:

$$\begin{bmatrix} -7 & 2 & -3 & 0 \\ 2 & -4 & -6 & 0 \\ -1 & -2 & -5 & 0 \end{bmatrix}$$

After a bunch of reductions,

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 + 2x_3 = 0 \end{cases}$$

If x_3 is nonzero, let $x_3 = c$. Then, we have set values for x_1, x_2.
$X = c(\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix})$ is always an eigenvector for $\lambda_1 = 5$.

For $\lambda_2 = \lambda_3 = -3$,

$$A - (-3I) = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Make the augemented matrix:

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -6 & 0 \\ -1 & -2 & 3 & 0 \end{bmatrix}$$

Add the first row to the third row, and subtract 2 times the first row from the second row.

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 2x_2 - 3x_3 = 0$. an equation of 3 unknowns.

Write $x_1 = 3x_3 - 2x_2$ with 2 free indep var and 1 dep var. We set the free variables to a value and see what happens.
If $x_2 = 1, x_3 = 0$ then $x_1 = -2$.
so $x^{(2)} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$
If $x_3 = 1, x_2 = 0$ then $x^{(3)} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$.

We get 2 linearly ind eigenvectors for the eigenvalue -3.

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6.1 Intro Fourier

Functions could be very complicated Example: $ln(1 + \sqrt{2sinx}), \tan(e^x - 1)$

We like to use simple functions to represent or approximate complicated functions.

Polynomials are easy to evaluate, differentiate, and integrate. We like to use polynomials to represent or aprrroximate "complicated" functions as studied in calculus.

The Taylor polynomial of degree n for a function $f(x)$ at $x = a$ is

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$T_n(a) = f(a), \quad T_n'(a) = f'(a), \quad T_n''(a) = f''(a), \ \ldots, \ T_n^{(n)}(a) = f^{(n)}(a)$

When $n = 1$, $y = T_1(x) = f(a) + f'(a)(x - a)$ is the tangent line of $f(x)$ at $(a, f(a))$.

He draws better and better approximations for f(x) using taylor series.

$T_n(x)$ approximates f(x) well when x is close to a.

Let $n \to \infty$.

$$\text{Taylor series} = T_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Example: If $f(x) = e^x$ and $a = 0$, then $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ because $e^0 = 1$ and the derivates of e^x is e^x. Other simple functions are sine and cosine functions.

We like to use sine, cosines,

$$1, cosx, sinx, cos(2x), sin(2x), \ldots, cos(nx), sin(nx).$$

to approximate f(x). Since the above sine and cos are 2π-periodic, assume that f(x) is 2π-periodic. We want to have

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n cos(nx) + b_n sin(nx))$$

for appropriate coefficients $a_0, a_1, b_1, \ldots, a_n, b_n, \ldots$

Lemma: For two integers $m, n \geq 1$, we have.

1)

$$\int_{-\pi}^{\pi} sinmx * cosnxdx = 0 \quad \text{for any} \quad m, n \geq 1$$

2)

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \begin{cases} \pi, & m = n \\ 0, & m \neq n \end{cases}$$

3)

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} \pi, & m = n \\ 0, & m \neq n \end{cases}$$

Proof for 1)

$$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$$

If $m = n$,

$$\int_{-\pi}^{\pi} \frac{1}{2}\left(\sin(mx+nx) + \sin(mx-nx)\right) = \frac{1}{2}\int_{-\pi}^{\pi}\left(\sin(mx+nx) + \sin(mx-nx)\right) = \frac{1}{2}\int_{-\pi}^{\pi}\left(\sin(2mx) + \sin(0)\right)$$

because $mx = nx$.

$$= \frac{1}{2}\int_{-\pi}^{\pi}\left(\sin(2mx) + \sin(0)\right) = \frac{1}{2}\int_{-\pi}^{\pi}\left(\sin(2mx) + 0\right) = \frac{1}{2}\int_{-\pi}^{\pi}\left(\sin(2mx)\right)$$

$$= \frac{1}{2*2m}\left(-\cos(2mx)\right)\Big|_{-\pi}^{\pi} = \frac{1}{2*2m}\left(-\cos(2m\pi) - (-\cos(2m(-\pi)))\right) = \frac{1}{2*2m}(0-0) = 0$$

Proof of 2)

$$\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right)$$

then equivalently, we have $\int_{-\pi}^{\pi}\frac{1}{2}\left(\cos(\alpha+\beta)+\cos(\alpha-\beta)\right)$

If $m = n$, the $cos(m-n)x = cos(0) = 1$.

$$\int_{-\pi}^{\pi}\frac{1}{2}\left(\cos(2mx)+1\right)dx = \frac{1}{2}\int_{-\pi}^{\pi}\cos(2mx)\,dx + \frac{1}{2}\int_{-\pi}^{\pi}1\,dx = \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot(2\pi) = \pi$$

The step where we get the 0 is from $sin(2m\pi) = 0$ for any m integer value.

If $m \neq n$,

$$\int_{-\pi}^{\pi}\frac{1}{2}\left(\cos((m+n)x)+\cos((m-n)x)\right)dx = \frac{1}{2}\int_{-\pi}^{\pi}\cos((m+n)x)dx + \frac{1}{2}\int_{-\pi}^{\pi}\cos((m-n)x)dx$$

$\int_{-\pi}^{\pi}\cos(kx\pi)dx = 0$ for any nonzero integer k from the sine function at $kx\pi$ being 0. Therefore, the whole expression is zero:

$= 0$

3 is not shown Now, we aim to find all coefficients in

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n cos(nx) + b_n sin(nx))$$

. To find a_0, integrate each term in the equation from $-\pi \quad \text{to} \quad \pi$.

$$\int_{-\pi}^{\pi} f(x)dx = 2\pi a_0 + \sum_{n=1}^{\infty}(a_n \int_{-\pi}^{\pi} cosnxdx + b_n \int_{-\pi}^{\pi} sinnxdx) = 2\pi a_0 + 0$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx =$$

To find a_m for $m \geq 1$, we multiply the equation (1) by cosmx.

$$f(x)cos(mx) = a_0 cos(mx) + cos(mx)\sum_{n=1}^{\infty}(a_n cos(nx) + b_n sin(nx))$$

and integrate the equation from $-\pi to \pi$.

$$\int_{-\pi}^{\pi} f(x)cosmxdx = a_0 \int_{-\pi}^{\pi} cosmxdx + \sum_{n=1}^{\infty}(a_n \int_{-\pi}^{\pi} cosnxcosmxdx)$$

, see the equations pi or 0.

$$= a_m * \pi + b_n \int_{-\pi}^{\pi} sinnxcosmxdx = a_m * \pi + 0$$

which is 0 by our previous calculations.

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(mx)dx$$

, $m \geq 1$.

To find b_m, we multiply the equation (1) by $sin(mx)$ and integrate it from $-\pi to \pi$ which yields

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(mx)dx$$

, for $m \geq l$, by property (3).

Theorem: Fourier Series : Under a mild condition (don't know what this mean), a 2π periodic function f(x) can be expanded into a Fourier Series as follows

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx)$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(nx)dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(nx)dx$$

for $n \geq 1$.

Example: Find the fourier series of a 2 pi periodic function given by

$$f(x) = \begin{cases} 1, & 0 < x < \pi \\ x, & -\pi < x < 0 \end{cases}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2}\left(1 - \frac{\pi}{2}\right)$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \\ &= \frac{1}{\pi} \left(\int_{-\pi}^{0} x \cos(nx)\, dx + \int_{0}^{\pi} \cos(nx)\, dx \right) \\ &= \frac{1}{\pi} \left[x \frac{\sin(nx)}{n} \Big|_{-\pi}^{0} - \int_{-\pi}^{0} \frac{\sin(nx)}{n}\, dx + \frac{\sin(nx)}{n} \Big|_{0}^{\pi} \right] \\ &= \frac{1}{\pi} \left(0 - 0 + \frac{\cos(nx)}{n^2} \Big|_{-\pi}^{0} + 0 \right) \\ &= \frac{1}{\pi} \left(\frac{1 - \cos(nx)}{n^2} \right) \\ &= \frac{1}{\pi} \left(\frac{1 - (-1)^n}{n^2} \right) \end{aligned}$$

$$b_n = 1/pi \int_{-\pi}^{\pi} f(x) sinnxdx$$

$$= \frac{1}{\pi} \left[\frac{\sin nx}{n} \Big|_{0}^{\pi} - \int_{0}^{\pi} \frac{\sin nx}{n} dx \right] + \frac{\sin nx}{n} \Big|_{0}^{\pi}$$

$$= \frac{1}{\pi} \left[0 - 0 + \frac{1 - \cos n\pi}{n^2} \right]$$

For $n = 1, \cos n\pi = -1$
For $n = 2, \cos n\pi = +1$
For $n = 3, \cos n\pi = -1$
For $n = 4, \cos n\pi = +1$

$$\cos(n\pi) = (-1)^n$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} x \sin nx\, dx + \int_{0}^{\pi} 1 \cdot \sin nx\, dx \right]$$

$$= \frac{1}{\pi} \left[x \left(-\frac{\cos nx}{n} \right) \Big|_{-\pi}^{0} - \int_{-\pi}^{0} \left(-\frac{\cos nx}{n} \right) dx \right]$$

$$= \frac{1}{\pi} \left[0 + \frac{\pi}{n} - \frac{\cos nx}{n^2} \Big|_{-\pi}^{0} \right]$$

$$= \frac{1}{\pi} \left[0 + \frac{\pi}{n} + \frac{\sin nx}{n^2} \Big|_{-\pi}^{0} \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi - 1 \cdot (-1)^n + 1}{n} \right]$$

$$b_n = \frac{(\pi - 1)(-1)^n + 1}{\pi n}$$

The Fourier series for the given function is

$$f(x) = \frac{1}{2}\left(1 - \frac{\pi}{2}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{\pi}\cdot\frac{1-(-1)^n}{n^2}\cos(nx) + \frac{1}{\pi}\cdot\frac{(\pi-1)(-1)^n+1}{n}\sin(nx)\right)$$

6.2 Odd and Even Functions

If $f(x)$ is an even function on $[-L, L]$, that is, $f(-x) = f(x)$ for $-L \leq x \leq L$, then

$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx.$$

If $f(x)$ is an odd function on $[-L, L]$, that is, $f(-x) = -f(x)$ for $-L \leq x \leq L$, then

$$\int_{-L}^{L} f(x)\,dx = 0.$$

Example: Find the Fourier Series of a 2 pi periodic function given by

$$f(x) = x^2 \quad \text{for} \quad -\pi < x < \pi$$

Okay, he really means $-pi + 2\pi n < x < pi + 2\pi n$ but it's all the same.

$$a_0 = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)dx$$

which is even and even so the resulting function is even. (Pf?)

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2\cos(nx)dx$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} x^2\cos(nx)\,dx \\
&= \frac{2}{\pi}\left[x^2\frac{\sin(nx)}{n}\Big|_0^\pi - \int_0^\pi 2x\frac{\sin(nx)}{n}dx\right] \\
&= \frac{4}{n\pi}\left[x\frac{\cos(nx)}{n}\Big|_0^\pi - \int_0^\pi \frac{\cos(nx)}{n}dx\right] \\
&= \frac{4}{\pi}\left[\frac{\pi(-1)^n}{n^2} - \frac{\sin(nx)}{n^2}\Big|_0^\pi\right] \\
&= \frac{4}{\pi}\cdot\frac{\pi(-1)^n}{n^2} \\
&= \frac{4(-1)^n}{n^2}
\end{aligned}$$

For b_n, it's an even times an odd function which is odd, and we have 0 as the integral.
The fourier series for $x^2, -\pi < x < \pi$

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx)$$

When $x = \frac{\pi}{2}$, we have an estimate

$$\frac{\pi^2}{4} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos\left(\frac{n\pi}{2}\right)$$

When $n = 2m$ is even, $\cos\left(\frac{n\pi}{2}\right) = \cos(m\pi) = (-1)^m$.
If n is odd, i.e., $n = 2m+1$, then $\cos\left(\frac{(2m+1)\pi}{2}\right) = 0$.
So,

$$\frac{\pi^2}{4} = \frac{\pi^2}{3} + \sum_{m=1}^{\infty} \frac{4}{(2m)^2}(-1)^m$$

or equivalently,

$$\pi^2 = 12\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots\right)$$

We also have a nice inequality:

$$\pi^2 - 12\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots \pm \frac{1}{n^2}\right) \leq \frac{1}{(n+1)^2}$$

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He restates the definition of fourier series expansion we have above.

Now, assume that f(x) is a periodic function of period $= p = 2L$. Where $L > 0$. Let

$$g(x) = f(\frac{Lx}{\pi})$$

Claim: g(x) is 2π periodic. Proof:

$$g(x+2\pi) = f(\frac{Lx + 2L\pi}{\pi}) = f(\frac{Lx}{\pi} + 2L) = f(\frac{Lx}{pi}) = g(x) \quad \text{because f is 2L periodic}$$

□

g(x) can now be expanded into

$$g(x) = a_0 + \sum_{n=1}^{\infty}(a_n cos(nx) + b_n sin(nx))$$

with a_0, a_n, b_n given by the integral formulas with respext to g.

$$f(\frac{Lx}{\pi}) = a_0 + \sum_{n=1}^{\infty}(a_n cos(nx) + b_n sin(nx))$$

Let $t = \frac{Lx}{\pi}$ or $x = \frac{\pi t}{L}$

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n cos(\frac{n\pi}{L}t) + a_n sin(\frac{n\pi}{L}t))$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)dx$$

Let $x = \frac{\pi t}{L}$, then $dx = \frac{\pi}{L}dt$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)dx = \frac{1}{2\cancel{\pi}}\int_{-L}^{L} g(\frac{\pi t}{L})\frac{\cancel{\pi}}{L}dt$$

Since $g(x) = f(\frac{Lx}{\pi})$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)cos(nx)dx$$

Using the same substitution,

$$= \frac{1}{L}\int_{-L}^{L} f(t)cos(\frac{n\pi}{L}t)dt$$

Similarly,

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)cos(\frac{n\pi}{L}t)dt$$

7.1 Periodic functions of period 2L

Theorem: Periodic 2L functions If f(x) is a periodic function of period $p = 2L$ where $L > 0$, then f(x) has the following fourier (trigonometric) series expansion.

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n cos(\frac{n\pi x}{L}) + b_n sin(\frac{n\pi x}{L}))$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)cos(\frac{n\pi}{L}x)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)sin(\frac{n\pi}{L}x)dx$$

Note: we swapped the t with x in the formula, but everything is equivalent.

Example : Find the fourier Series expansion for f(x) which has a period $p = 2.$ and is defined by

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ -1, & -1 < x < 0 \end{cases}$$

f(x) is an odd function with $L = 1$, since $p = 2L = 2$.
Thus,

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)dx = 0$$

.

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)cos(\frac{n\pi}{1}x)dx = 0$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)sin(\frac{n\pi}{1}x)dx$$

$$= 2\int_{0}^{1} f(x)sin(\pi x)dx$$

$$= 2\int_{0}^{1} sin(\pi x)dx$$

$$= 2(-\frac{cos(n\pi x)}{n\pi})\Big|_0^1$$

$$= 2(\frac{1-(-1)^n}{n\pi})$$

Hence,

$$f(x) = \sum_{n=1}^{\infty} 2(\frac{1-(-1)^n}{n\pi} sin(n\pi x))$$

When $n = 2m$ or even, $(-1)^{2m} = 1$,
$\frac{1-(-1)^n}{2m\pi} = 0$.

If $n = 2m + 1$ is odd,
$\frac{1-(-1)^n}{(2m+1)\pi} = \frac{2}{(2m+1)\pi}$

$$f(x) = \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi} sin((2m+1)\pi x)$$

Note: m = 0 because we now start at 1 for all values 2m+1.

7.2 Sine and cosine expansions on the half range

Suppose f(x) is defined on [0,L]

f(x) can be extended to the other side to be an even function $f_e(-x) = f_e(x)$ on [-L,L] by setting

$$f_e(x) = \begin{cases} f(x), & 0 < x < L \\ f(-x), & -L < x < 0 \end{cases}$$

Then we can use our Fourier expansions.

$$b_n = \frac{1}{L}\int_{-L}^{L} f_e(x) sin(\frac{\pi n}{L}x)dx = 0$$

because sin is odd and $f_e(x)$ is even by definition. We have a cosine exapnsion for $f_e(x)$ over [-L,L]

$$f_e(x) = a_0 + \sum_{n=1}^{\infty} a_n cos(\frac{n\pi}{L}x)$$

In particular, f(x) has a cosine expansion in [0,L]

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n cos(\frac{n\pi}{L}x)$$

where

$$a_0 = \frac{\not{2}}{\not{2}L}\int_0^L f(x)dx$$

$$a_n = \frac{2}{L}\int_0^L f(x) cos(\frac{n\pi}{L}x)dx$$

Similarly, f(x) can be extended to an odd function $f_o(x)$ in [-L,L] by letting

$$f_o(x) = \begin{cases} f(x), & 0 < x < L \\ -f(x), & -L < x < 0 \end{cases}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f_o(x)dx = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f_0(x) cos(\frac{n\pi}{L}x)dx = 0$$

hence, $f_o(x)$ has a sine expansion in [-L,L]

$$f_o(x) = \sum_{n=1}^{\infty} b_n sin(\frac{n\pi}{L}x)$$

$$f(x) = \sum_{n=1}^{\infty} b_n sin(\frac{n\pi}{L}x)$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)sin(\frac{n\pi}{L}x)dx$$

Note: The even and odd stuff is a little unintuitive. What we're doing is taking a function that defined for some range [0,L]. Then, we append the other side and use fourier series to find the fourier series of the new function. That new function is correct for [0,L].

Example: Find both sine and cosine expansions of f(x) = x in [0,1]. Extend f(x) = x in [0,1] to an even function on [-1,1]

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ -x, & -1 < x < 0 \end{cases} = \begin{cases} |x|, & -1 < x < 1 \end{cases}$$

$L = 1$

$$a_n = \frac{2}{L}\int_0^L f(x)cos(\frac{n\pi x}{L})dx = 2\int_0^1 xcos(n\pi x)dx = 2(\frac{(-1)^n - 1}{(n\pi)^2})$$

For $0 < x < 1$,

$$f(x) = x = \frac{1}{2} + 2\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{(n\pi)^2} cos(n\pi x)$$

For sine expansion, we extend f(x) = x in [0,1] to an odd function in [-1,1] by setting

$$f_o(x) = \begin{cases} x, & 0 < x < 1 \\ -(-x), & -1 < x < 0 \end{cases}$$

or just x itself (because x is odd).

$a_0 = a_n = 0$ and

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)\sin\Big(\frac{n\pi}{L}x\Big)\,dx \\ &= 2\int_0^1 x\sin\Big(\frac{n\pi}{1}x\Big)\,dx \\ &\overset{\text{IBP}}{=} 2\left[\frac{-x\cos(n\pi x)}{n\pi}\right]_0^1 + \frac{2}{n\pi}\int_0^1 \cos(n\pi x)\,dx \\ &= \frac{2(-1)^{n+1}}{n\pi}. \end{aligned}$$

Hence, for $0 < x < 1$,

$$f(x) = x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} sin(n\pi x)$$

See figure 1 below.

Tomorrow: Sample final exam problems

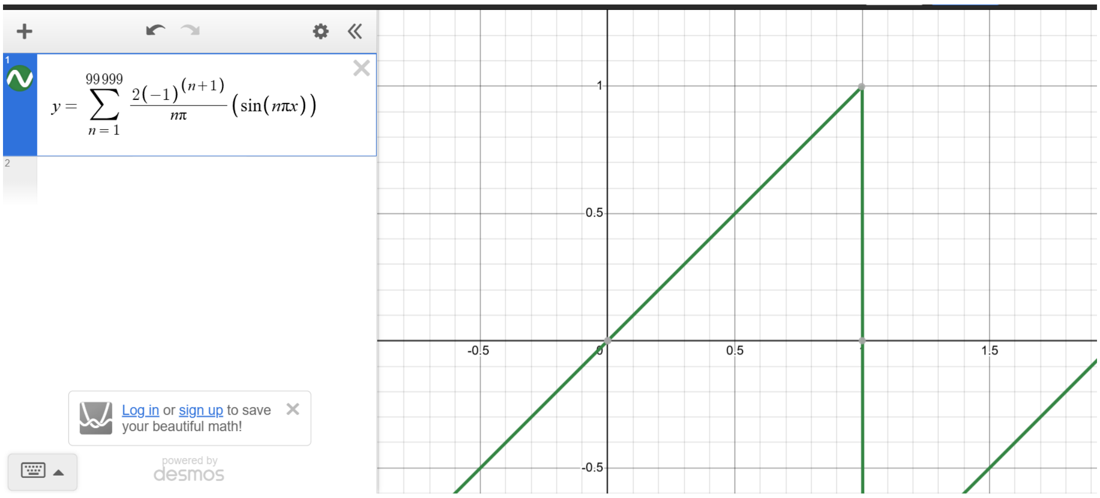

Figure 1: A graph of the Fourier Series for y = x from 0 to 1.

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8 8/13 Example Problems

8.1 Example Final: Questions

1. Solve the equation
$$e^y y' = x\sec(2y)$$

2. Is
$$(2x\cos y + 1)dx - x^2 \sin y\, dy = 0$$
an exact equation? If yes, find its general solution

3. Find a general solution of
$$y' + 2y = x$$

4. Solve the IVP:
$$x^2y'' + 3xy' - 5y = 0$$
,
$$y(1) = -2, y'(1) = 1$$

5. Find a real general solution of
$$y'' - 2y' + 10y = 0$$

6. Solve
$$y'' - 6y' + 9y = x^2$$
by the method of undetermined coefficients

7. Use the variation of parameters to solve
$$x^2y'' - xy' - 3y = x^5 - x^3$$

8. Solve
$$\begin{cases} y_1' = 3y_1 + 2y_2 \\ y_2' = -2y_1 - 2y_2 \end{cases}$$

9. Find a general solution of
$$y^{(4)} + 3y'' - 4y = 0$$

10. Use Laplace transforms to solve the IVP:
$$y'' - y' - 6y = 0$$
,
$$y(0) = 1, y'(0) = -1$$

11. Find the Fourier series of a 2π-periodic function given by
$$f(x) = \begin{cases} 1, & 0 \le x < \pi \\ -1, & -\pi < x < 0 \end{cases}$$

12. Use Gauss elimination methods to solve
$$\begin{cases} x + y + 2z = 3 \\ 2x - y - z = 4 \\ -x + 2y + z = -3 \end{cases}$$

13(a) Find the inverse of the matrix
$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

13(b) Use A^{-1} in (a) to solve the system

$$\begin{cases} x - y + z = 3, \\ -x + z = 2, \\ y - z = -5. \end{cases}$$

14 Use Cramer's rule to solve

$$\begin{cases} 3x - 2y = 5, \\ 4x + 5y = -6. \end{cases}$$

15 Calculate

$$\det(A) = \begin{vmatrix} 2 & -1 & 0 & 1 \\ -1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & -2 & 1 \end{vmatrix}.$$

8.2 Example Final: Answers

1. Solve the equation

$$e^y y' = x\sec(2y)$$

Answer

$$\begin{aligned} e^{-x}y' = e^{-x}\frac{dy}{dx} &= x\sec(2y) \\ \rightarrow e^{-x}dy &= dx * x * \sec(2y) \\ \rightarrow \frac{dy}{sec(2y)} &= xe^x dx \end{aligned}$$

Now, use that $sec = \frac{1}{cos}$,

$$\begin{aligned} cos(2y)dy &= xe^x dx \\ \int cos(2y)dy &= \int xe^x dx \\ \frac{sin(2y)}{2} + C = xe^x - \int 1 * e^x dx &= xe^x - e^x + C \\ \boxed{\frac{sin(2y)}{2} = (x-1)e^x + C} \end{aligned}$$

Which is good enough, and we don't run into bounds issues on y or x (which we have for arcsin).

2. Is

$$(2x\cos y + 1)dx - x^2 \sin y\, dy = 0$$

an exact equation? If yes, find its general solution

Answer

Check for exactness:

$$\frac{\partial}{\partial y}(2x\cos y+1)=-2x\sin y$$

$$\frac{\partial}{\partial x}(-x^2\sin y)=-2x\sin y$$

Yes, exact.

There exists u(x,y) such that

$$du=\frac{\partial u}{\partial x}\,dx+\frac{\partial u}{\partial y}\,dy=M dx+N dy$$

$$\int\frac{\partial u}{\partial x}\,dx=\int(2x\cos y+1)\,dx=x^2\cos y+x+k(y)=u(x,y)$$

$$\frac{\partial}{\partial y}(u(x,y))=\frac{\partial}{\partial y}\big(x^2\cos y+x+k(y)\big)=-x^2\sin y+0+k'(y)$$

$$-x^2\sin y+k'(y)=-x^2\sin y$$

$$k'(y)=0\quad\Rightarrow\quad k(y)=C$$

$$\boxed{u(x,y)=x^2\cos y+x=C}$$

3. Find a general solution of

$$y'+2y=x$$

Answer

The form of the differential equation is:

y' + p(x)y = r(x)

We solve

$$y'+2y=x.$$

Let $h=\int p\,dx=\int 2\,dx=2x$. Then

$$y=e^{-h}\Big(\int r(x)e^h\,dx+C\Big)=e^{-2x}\Big(\int xe^{2x}\,dx+C\Big).$$

$$\begin{aligned}\int xe^{2x}\,dx&=\frac{x}{2}e^{2x}-\int\frac{1}{2}e^{2x}\,dx &&\text{(IBP: } u=x,\ dv=e^{2x}dx)\\&=\frac{x}{2}e^{2x}-\frac{1}{4}e^{2x}+C\\ \Rightarrow\quad y&=e^{-2x}\Big(\Big(\frac{x}{2}-\frac{1}{4}\Big)e^{2x}+C\Big)=\boxed{\frac{x}{2}-\frac{1}{4}+Ce^{-2x}}.\end{aligned}$$

Teacher got a calculation wrong.

2nd Way (Writer's choice)

The form of the differential equation is: y' + p(x)y = r(x).
Find an integragrating factor I(x):

$$I(x)y' + I(x)p(x)y = I(x)r(x)$$

Assume the left side is (Iy)':

$$(I(x)y)' = I(x)y' + I(x)'y = I(x)\,y' + I(x)\,p(x)\,y$$

. Now,

$$I'\,y = I\,py$$

which is true if $I(x)' = I(x)\,p(x)$.

$$\frac{dI}{dx} = p(x)I(x)$$
$$\implies ln(I(x)) = \int p(x)dx$$
$$\implies I(x) = e^{\int p\,dx}$$

Plugging in,

$$(I(x)y)' = e^{\int p\,dx}r(x)$$
$$\text{Take an integral}$$
$$I(x)y = \int e^{\int p\,dx}r(x)dx$$
$$y = \frac{1}{I(x)}\int e^{\int p\,dx}r(x)dx$$

4. Solve the IVP:

$$x^2y'' + 3xy' - 5y = 0$$

,

$$y(1) = -2, y'(1) = 1$$

Answer

Euler-Cauchy ODE: Assume $y = x^m$

$$(m)(m-1)x^m + 3mx^m - 3x^m = 0$$

$$(m-1)(m) + 3m - 3 = 0$$
$$m^2 - m + 3m - 3 = 0$$
$$m^2 + (3-1)m - 3 = 0$$
$$m^2 + 2m - 3 = 0$$
$$(m+3)(m-1) = 0$$
$$m_1 = -3, \quad m_2 = 1$$

Hence,

$$y_1 = x^{-3}, \quad y_2 = x$$

General solution:

$$y = c_1 x^{-3} + c_2 x$$

If

$$y(1) = -2 \implies c_1(1)^{-3} + c_2(1) = -2$$

For

$$y' = -3c_1 x^{-4} + c_2$$

If

$$y'(1) = 1 \implies -3c_1(1) + c_2 = 1$$
$$c_1 + c_2 - (-3c_1 + c_1) = -2 - 1$$
$$4c_1 = -3 \implies c_1 = \frac{-3}{4}$$

Plug in to get:

$$c_2 = \frac{-5}{4}$$

Particular Solution:

$$\boxed{y_p = \frac{-3}{4}x^{-3} + \frac{-5}{4}x}$$

5. Find a real general solution of

$$y'' - 2y' + 10y = 0$$

Answer

Second order linear homogeneous ODE with constant coefficients: Characteristic equation:

$$\lambda^2 - 2\lambda + 10 = 0$$

$$\lambda = 1 \pm 3i \quad \text{using the quadratic equation.}$$

A general solution is:

$$\boxed{y = e^x(c_1 cos(3x) + c_2 sin(3x))}$$

6. Solve

$$y'' - 6y' + 9y = x^2$$

by the method of undetermined coefficients

Answer

Solving the homogeneous equation:

$$\begin{aligned} &y'' - 6y' + 9y = 0 \\ &\implies \lambda^2 - 6\lambda + 9 = 0 = (\lambda - 3)^2 \\ &\implies \lambda = 3 \quad \text{double root} \\ &y_h = e^{3x}(c_1 + c_2 x) \end{aligned}$$

Since $r(x) = x^2$, choose

$$y_p = k_2 x^2 + k_1 x + k_0$$

Then,

$$y' = 2k_2 x + k_1, y'' = 2k_2$$

Plug in:

$$\begin{aligned} y'' - 6y' + 9y &= x^2 \\ (2k_2) - 6(2k_2 x + k_1) + 9(k_2 x^2 + k_1 x + k_0) &= x^2 \\ 2k_2 - 12k_2 x - 6k_1 + 9k_2 x^2 + 9k_1 x + 9k_0 &= x^2 \\ 9k_2 x^2 + 9k_1 x - 12k_2 x + 9k_0 - 6k_1 + 2k_2 &= x^2 \end{aligned}$$

For the matching values of

$$x^2, x, 1$$

$$\begin{cases} 9k_2 = 1 \\ 9k_1 - 12k_2 = 0 \\ 9k_0 - 6k_1 + 2k_2 = 0 \end{cases} \implies \begin{cases} k_2 = \frac{1}{9} \\ k_1 = \frac{\frac{12}{9}}{9} = \frac{4}{27} \\ k_0 = \frac{\frac{24}{27} - \frac{2}{9}}{9} = \frac{2}{27} \end{cases}$$

$$y = y_h + y_p = \boxed{e^{3x}(c_1 + c_2 x) + \frac{1}{9}x^2 + \frac{4}{27}x + \frac{2}{27}}$$

7. Use the variation of parameters to solve

$$x^2 y'' - xy' - 3y = x^5 - x^3$$

Answer

Given the non-homogeneous Cauchy-Euler equation:

$$x^2y'' - xy' - 3y = x^5 - x^3$$

Step 1: Solve the homogeneous equation.

$$x^2y'' - xy' - 3y = 0$$

This is a Cauchy-Euler equation. We assume a solution of the form $y = x^m$. The characteristic equation is:

$$m(m-1) - m - 3 = 0$$
$$m^2 - m - m - 3 = 0$$
$$m^2 - 2m - 3 = 0$$
$$(m-3)(m+1) = 0$$

The roots are $m_1 = -1$ and $m_2 = 3$. The complementary solution is:

$$y_h = c_1y_1 + c_2y_2 = c_1x^{-1} + c_2x^3$$

Step 2: Use Variation of Parameters. First, we must normalize the equation by dividing by x^2:

$$y'' - \frac{1}{x}y' - \frac{3}{x^2}y = x^3 - x$$

So, the function $f(x) = x^3 - x$. The fundamental solutions are $y_1 = x^{-1}$ and $y_2 = x^3$. The Wronskian is:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^{-1} & x^3 \\ -x^{-2} & 3x^2 \end{vmatrix} = (x^{-1})(3x^2) - (x^3)(-x^{-2}) = 3x + x = 4x$$

We find the functions u and v for the particular solution $y_p = uy_1 + vy_2$.

$$u = -\int \frac{y_2 f(x)}{W}dx = -\int \frac{x^3(x^3-x)}{4x}dx = -\frac{1}{4}\int \frac{x^6 - x^4}{x}dx$$

$$u = -\frac{1}{4}\int (x^5 - x^3)dx = -\frac{1}{4}\left(\frac{x^6}{6} - \frac{x^4}{4}\right) = -\frac{1}{24}x^6 + \frac{1}{16}x^4$$

$$v = \int \frac{y_1 f(x)}{W}dx = \int \frac{x^{-1}(x^3-x)}{4x}dx = \frac{1}{4}\int \frac{x^2-1}{x}dx$$

$$v = \frac{1}{4}\int \left(x - \frac{1}{x}\right)dx = \frac{1}{4}\left(\frac{x^2}{2} - \ln|x|\right) = \frac{1}{8}x^2 - \frac{1}{4}\ln|x|$$

Step 3: Construct the particular and general solutions. The particular solution is $y_p = uy_1 + vy_2$:

$$y_p = \left(-\frac{1}{24}x^6 + \frac{1}{16}x^4\right)x^{-1} + \left(\frac{1}{8}x^2 - \frac{1}{4}\ln|x|\right)x^3$$

$$y_p = -\frac{1}{24}x^5 + \frac{1}{16}x^3 + \frac{1}{8}x^5 - \frac{1}{4}x^3\ln|x|$$

$$y_p = \left(-\frac{1}{24} + \frac{3}{24}\right)x^5 + \frac{1}{16}x^3 - \frac{1}{4}x^3\ln|x|$$

$$y_p = \frac{2}{24}x^5 + \frac{1}{16}x^3 - \frac{1}{4}x^3\ln|x| = \frac{1}{12}x^5 + \frac{1}{16}x^3 - \frac{1}{4}x^3\ln|x|$$

The general solution is $y = y_h + y_p$:

$$\boxed{y = c_1x^{-1} + c_2x^3 + \frac{1}{12}x^5 + \frac{1}{16}x^3 - \frac{1}{4}x^3\ln|x|}$$

8. Solve

$$\begin{cases} y_1' = 3y_1 + 2y_2 \\ y_2' = -2y_1 - 2y_2 \end{cases}$$

Answer

$$A = \begin{pmatrix} 3 & 2 \\ -2 & -2 \end{pmatrix}$$

Find eigenvalues by $det(A - \lambda I_2)$

$$\begin{vmatrix} 3-\lambda & 2 \\ -2 & -2-\lambda \end{vmatrix} = (\lambda-3)(\lambda+2) - (-4) = -2 - \lambda + \lambda^2 = (\lambda-2)(\lambda+1)$$

$\lambda_1 = -1, \lambda_2 = 2$

Eigenvectors: For $\lambda_1 = -1$,

$$(A - \lambda_1 I_2)X = 0$$

$$\begin{pmatrix} 4 & 2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Okay, when you multiply, it gives $x_2 = -2x_1$
Choose $x_1 = 1$ (it doesn't matter because we multiply by a scalar). Then, $x_2 = -2$. The eigenvector is

$$X^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

The other eigenvector (for $\lambda_2 = 2$) is found from $(A - 2I_2)X = 0$ with the same procedure.

$$X^{(2)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\boxed{y = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{2t}}$$

9. Find a general solution of

$$y^{(4)} + 3y'' - 4y = 0$$

Answer

$$\begin{aligned} \lambda^4 + 3\lambda^2 - 4 &= 0. \\ (\lambda^2 + 4)(\lambda^2 - 1) &= 0 \\ \lambda &= \pm 2i, \pm 1 \end{aligned}$$

Then, a general solution is

$$\boxed{y = c_1 cos(2x) + c_2 sin(2x) + c_3 e^{-x} + c_4 e^{x}}$$

10. Use Laplace transforms to solve the IVP:

$$y'' - y' - 6y = 0$$

,

$$y(0) = 1, y'(0) = -1$$

Answer

$$\begin{aligned} L(y'') - L(y') - 6L(y) &= L(0) \\ \big(s^2L(y) - sy(0) - y'(0)\big) - \big(sL(y) - y(0)\big) - 6L(y) &= 0 \\ \big(s^2L(y) - s(1) - (-1)\big) - \big(sL(y) - 1\big) - 6L(y) &= 0 \\ s^2L(y) - s + 1 - sL(y) + 1 - 6L(y) &= 0 \\ (s^2 - s - 6)L(y) &= s - 2 \\ L(y) &= \frac{s-2}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}. \end{aligned}$$

Partial Fractions:

$$s - 2 = A(s+2) + B(s-3)$$

$$\text{If } s = 3: \quad A = \frac{1}{5}$$

$$\text{If } s = -2: \quad B = \frac{4}{5}$$

Thus,

$$L(y) = \frac{1}{5} \cdot \frac{1}{s-3} + \frac{4}{5} \cdot \frac{1}{s+2}.$$

Inverse Laplace Transform: $L^{-1}L(y) = y$

$$\begin{aligned} y(t) &= \frac{1}{5}\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) + \frac{4}{5}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) \\ &= \boxed{\frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}}. \end{aligned}$$

11. Find the Fourier series of a 2π-periodic function given by

$$f(x) = \begin{cases} 1, & 0 \le x < \pi \\ -1, & -\pi < x < 0 \end{cases}$$

Answer

We see that the function is odd.

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0 \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = 0 \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\,dx \end{aligned}$$

If $f(x) = 1$ for $0 < x < \pi$, then:

$$b_n = \frac{2}{\pi}\int_0^{\pi}\sin(nx)\,dx = \frac{2}{\pi}\left[\frac{-\cos(nx)}{n}\right]_0^{\pi} = \frac{2}{\pi}\cdot\frac{1-(-1)^n}{n}.$$

$$\boxed{f(x) = \sum_{n=1}^{\infty}\frac{2}{\pi}\frac{1-(-1)^n}{n} sin(nx)}$$

12. Use Gauss elimination methods to solve

$$\begin{cases} x + y + 2z = 3 \\ 2x - y - z = 4 \\ -x + 2y + z = -3 \end{cases}$$

Answer

Augmented Matrix:

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & -1 & -1 & 4 \\ -1 & 2 & 1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & -1 & -1 & 4 \\ 0 & 1 & 1 & 0 \end{pmatrix} (R_3 + R_1)/3$$

$$\begin{pmatrix} 1 & 0 & 1 & 3 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{matrix} R_1 - R_3 \\ R_2 + R_3 \\ \end{matrix} \to \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix} R_1 - R_2 \to \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \end{pmatrix} R_3 - R_1$$

$$\boxed{x = 2, \quad y = -1, \quad z = 1}$$

13(a) Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

13(b) Use A^{-1} in (a) to solve the system

$$\begin{cases} x - y + z = 3, \\ -x + z = 2, \\ y - z = -5. \end{cases}$$

Answer

13(a)

There are many ways to find the inverse.
He uses the method where you append the identity matrix.

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right) R_2 + R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) R_3 + R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) -(R_2 - 2R_3)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) R_1 + R_2 - R_3$$

$$\boxed{A^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}}$$

13(b)
Write the matrix representation: $\mathbf{Ax} = \mathbf{b}$

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$$

$$\textbf{Use} \quad A^{-1}Ax = Ix = x = A^{-1}b$$

: Or

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \\ 0 \end{bmatrix}$$

$$\boxed{x = -2, \quad y = -5, \quad z = 0}$$

14 Use Cramer's rule to solve

$$\begin{cases} 3x - 2y = 5, \\ 4x + 5y = -6. \end{cases}$$

Answer

$$\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = 23 \neq 0$$

$$x = \frac{\begin{vmatrix} 5 & -2 \\ -6 & 5 \end{vmatrix}}{23} = \boxed{\frac{13}{23}}$$

$$y = \boxed{\frac{-38}{23}} \quad \text{by plugging in or by Cramer}$$

15 Calculate

$$\det(A) = \begin{vmatrix} 2 & -1 & 0 & 1 \\ -1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & -2 & 1 \end{vmatrix}$$

Answer

$$= \begin{vmatrix} 2 & -1 & 1 & 1 \\ -1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & -2 & 1 \\ & & C_3 - C_2 & \end{vmatrix}$$

Use the 3rd row:

$$= 0 + (-1)^{3+2=5} 1 * \begin{vmatrix} 2 & 1 & 1 \\ -1 & 1 & 2 \\ 2 & -2 & 1 \end{vmatrix} + 0 + 0$$

$$= - \begin{vmatrix} 2 & 3 & 1 \\ -1 & 0 & 2 \\ 2 & 0 & 1 \\ & C_2 + C_1 & \end{vmatrix}$$

Use the second column:

$$= -(-1)^{1+2=3} 3 \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= 3((-1) - 4) = 3 \times (-5) = \boxed{-15}$$

9 8/14

Question and answer. There was a correction.

$$A^{-1} = \frac{1}{det(A)} \cdot \text{adj}(A)$$

It's corrected in the notes now.